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THEORY OF POROELASTIC PLATES WITH IN-PLANE DIFFUSION

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Abstract—A theory is developed for fluid-saturated poroelastic plates made of a material for which diffusion is possible in the in-plane directions only. Both bending and in-plane loading are considered. The plates are isotropic in the plate plane and obey the Kirchhoff hypotheses. Biot's constitutive law is adopted and Darcy's law is used to describe the fluid flow in pores. Quasi-static and transverse vibration problems are investigated. Example solutions are presented and observations are made on the types of deflection/vibration patterns exhibited. © 1997 Elsevier Science Ltd.

1. INTRODUCTION

The use of the poroelastic model is widely reported in the literature for massive structures as in geomechanical problems, for which the poroelastic bulk material (e.g., soil or rock) contains small pores which are connected and filled with a fluid (e.g., water or oil). Diffusion occurs, i.e. the fluid flows in the pores, when pore pressure gradients are produced by nonuniform volume strain in the pores of the medium. The process is generally time dependent due to the viscosity of the pore fluid. For light structures, such as porous rods, plates and shells, much less work has been reported. The papers that do appear in the literature are generally motivated by biomechanical problems, e.g. Nowinski and Davis (1972), Taber (1992) and Zhang and Cowin (1994). It has been reported that such poroelastic models are helpful in studying the mechanical response of biological tissues and organs such as bone, hearts and bladders. When such tissue elements are modeled as isotropic bending elements, the diffusion is predominantly in the transverse direction, i.e. perpendicular to the plate plane, due to greater stress gradients in this direction, while the diffusion is negligible in the longitudinal direction(s). This is evinced in the above references, and in Theodorakopoulos and Beskos (1993, 1994) which consider the poroelastic plate model as applied to geomechanical as well as to biomechanical problems.

The opposite situation is considered here, i.e. the micro-structure of the material is such that diffusion is predominantly or only in the longitudinal direction(s) while it can be considered negligible in the transverse direction. We are motivated by the mechanical behaviours of such elements as plant stems, but it is readily evident that it is possible to manufacture such artificial materials. In Li *et al.* (1995) the relevant equations are derived for poroelastic rods and solutions are found for quasi-static bending. In Li *et al.* (1996), flexural vibrations of beams are investigated and examples of forced and free vibration solutions are given.

In the present work, we consider poroelastic plates made of a material which is isotropic in the plate plane and for which diffusion is possible in the longitudinal directions (in-plane directions) only. The Kirchhoff plate theory is employed and Biot's three dimensional theory of poroelasticity is adapted to the present case. The basic equations are so derived that they could be easily extended for the situation of an orthotropic poroelastic plate. In-plane boundary loads are also considered. Closed form solutions are found for the quasi-static case and for vibrations for several problems. Some unique features of the time dependent behaviours for such plates are pointed out.

2. BASIC EQUATIONS FOR A THIN PLATE

The constitutive equations for a transversely isotropic poroelastic material are given, according to Biot (1962), as

$$\begin{aligned}
\tau_{11} &= 2B_1\varepsilon_{11}+B_2(\varepsilon_{11}+\varepsilon_{22})+B_3\varepsilon_{33}+B_6\zeta \\
\tau_{22} &= 2B_1\varepsilon_{22}+B_2(\varepsilon_{11}+\varepsilon_{22})+B_3\varepsilon_{33}+B_6\zeta \\
\tau_{33} &= B_4\varepsilon_{33}+B_3(\varepsilon_{11}+\varepsilon_{22})+B_7\zeta \\
\tau_{12} &= 2B_1\varepsilon_{12}, \tau_{23} = 2B_5\varepsilon_{23}, \tau_{31} = 2B_5\varepsilon_{31} \\
p_f &= B_6(\varepsilon_{11}+\varepsilon_{22})+B_7\varepsilon_{33}+B_8\zeta
\end{aligned} \tag{1}$$

where the material is taken to be isotropic in the x_1x_2 plane. B_m $(m = 1\text{–}8)$ are the material constants introduced by Biot. τ_{ij} are the total stresses of the bulk material $(i,j = 1, 2, 3)$, i.e. the stresses averaged over both the solid and fluid phases, and ε_{ij} are the strains of the solid skeleton. p_f is the pore pressure and ζ the fluid content increment which can be calculated by

$$\zeta = \phi(\varepsilon_{ii}-\varepsilon^f_{ii}) \tag{2}$$

where ϕ is the porosity and ε^f_{ij} is the strain of the fluid. In this paper a repeated index refers to a free index and the summation convention is adopted, except when specifically noted.

Eliminating ζ from the first three equations of (1) by using the last of these equations, we obtain the constitutive equations in the form

$$\begin{Bmatrix}\tau_{11}\\ \tau_{22}\\ \tau_{33}\end{Bmatrix} = \begin{bmatrix}\hat{B}_{11} & \hat{B}_{12} & \hat{B}_{13}\\ \hat{B}_{12} & \hat{B}_{11} & \hat{B}_{13}\\ \hat{B}_{13} & \hat{B}_{13} & \hat{B}_{33}\end{bmatrix}\begin{Bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \varepsilon_{33}\end{Bmatrix} - \begin{Bmatrix}\alpha_1\\ \alpha_1\\ \alpha_3\end{Bmatrix}p_f \tag{3}$$

and

$$\begin{aligned}
&\tau_{12} = 2G\varepsilon_{12}, \quad \tau_{23} = 2G_3\varepsilon_{23}, \quad \tau_{31} = 2G_3\varepsilon_{31} \\
&p_f = F[\zeta-\alpha_1(\varepsilon_{11}+\varepsilon_{22})-\alpha_3\varepsilon_{33})]
\end{aligned} \tag{4}$$

where the material constants introduced here can be expressed in terms of Biot's constants through

$$\begin{aligned}
&\hat{B}_{11} = 2B_1+B_2-B_6^2/B_8 \quad \hat{B}_{12} = B_2-B_6^2/B_8 \\
&\hat{B}_{13} = B_3-B_6B_7/B_8 \quad \hat{B}_{33} = B_4-B_7^2/B_8 \\
&\alpha_1 = -B_6/B_8 \quad \alpha_3 = -B_7/B_8 \\
&F = B_8 \quad G_3 = B_5 \quad G = (\hat{B}_{11}-\hat{B}_{12})/2.
\end{aligned} \tag{5}$$

Among the above, G can be considered as a derived parameter so that only 8 independent material constants are involved.

We consider a Kirchhoff plate with an arbitrary boundary made of the material defined above. The x_1x_2 plane is taken to be the mid-plane of the plate, shown as in Fig. 1. Letting τ_{33} be zero, we have

$$\varepsilon_{33} = [\alpha_3p_f-\hat{B}_{13}(\varepsilon_{11}+\varepsilon_{22})]/\hat{B}_{33} \tag{6}$$

and thus ε_{33} can be eliminated from the first and second equations of (3), and hence the constitutive equations for the plate can be simplified to

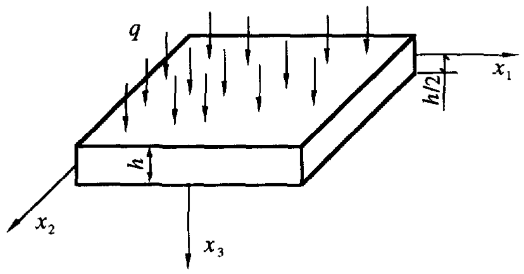

Fig. 1. A rectangular plate with dimension $L_1 \times L_2 \times h$ ($h \ll L_1$ and $h \ll L_2$) subjected to distributed normal load.

$$\tau_{\alpha\beta} = D_{\alpha\beta\gamma\delta}\varepsilon_{\gamma\delta} - \eta p_f \delta_{\alpha\beta}$$
$$\zeta = \hat{\beta} p_f + \eta \varepsilon_{\alpha\alpha} \tag{7a,b}$$

α, β and γ, $\delta = 1, 2$ and the stress-strain relations for τ_{23} and τ_{31} are omitted because the transverse shear strains are expected to be negligible for the thin plate; also it is not necessary to include the third equation of (3). $\delta_{\alpha\beta}$ is the Kronecker delta, i.e. $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ otherwise. Here the stiffness coefficients for the plate are

$$D_{1111} = D_{2222} = \hat{D}, \quad D_{1122} = v\hat{D}$$
$$D_{1212} = \hat{D}(1-v)/2, \quad D_{1112} = D_{1222} = 0 \tag{8}$$

and

$$\hat{\beta} = \frac{1}{F} + \frac{\alpha_3^2}{\hat{B}_{33}}, \quad \eta = \alpha_1 - \frac{\hat{B}_{13}}{\hat{B}_{33}}\alpha_3 \tag{9}$$

where $\hat{D} = \hat{B}_{11} - \hat{B}_{13}^2/\hat{B}_{33}$ and $v = (\hat{B}_{12} - \hat{B}_{13}^2/\hat{B}_{33})/(\hat{B}_{11} - \hat{B}_{13}^2/\hat{B}_{33})$. Those elements of the fourth-order tensor $D_{\alpha\beta\gamma\delta}$ which are not given in (8) are defined by the following symmetry relations

$$D_{\alpha\beta\gamma\delta} = D_{\alpha\beta\delta\gamma} = D_{\beta\alpha\gamma\delta} = D_{\gamma\delta\alpha\beta}. \tag{10}$$

Thus we see that in our situation a total of four parameters, $\hat{D}$, v, $\hat{\beta}$ and η, have so far been involved; these are determined by the seven independent material constants given in (5). We now consider the physical meaning of the parameters. We see from (7a) that the $D_{\alpha\beta\gamma\delta}$, which involve $\hat{D}$ and v, are the stiffness coefficients of the plate when the pore fluid is drained or when the pore pressure is zero. Further from (7b) we see that η is the ratio of the pore volume change to the sum of the two normal strains (area strain) within the plane, again in the same situation. Thus $D_{\alpha\beta\gamma\delta}$ and η depend only on the elastic properties and the microgeometry of the solid skeleton. On the other hand, $\hat{\beta}$ depends also on the fluid compressibility; it can be interpreted as the fluid content increment when the pore pressure equals 1 and the area strain in the plane is zero. Furthermore, if we consider a tensile test for the drained plate, i.e., let τ_{22} be zero, we would have $\varepsilon_{22} = -v\varepsilon_{11}$ and $\tau_{11} = \hat{D}(1-v^2)\varepsilon_{11}$. Hence v is the Possion's ratio and $\hat{E} = \hat{D}(1-v^2)$ is the Young's modulus for the drained case.

Now consider the geometrical relations for the plate

$$\varepsilon_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) - w_{,\alpha\beta}x_3 \tag{11}$$

where u_α is the displacement at the mid-plane of the plate in the x_α direction and w is the perpendicular deflection of the mid-plane. $u_{\alpha,\beta}$ refers to the partial derivative of u_α with respect to x_β, and so on. These relations are the same as those for a thin elastic plate.

Taking notice of the symmetry of the tensor $D_{\alpha\beta\gamma\delta}$, eqn (7a) can then be rewritten by (11) as follows

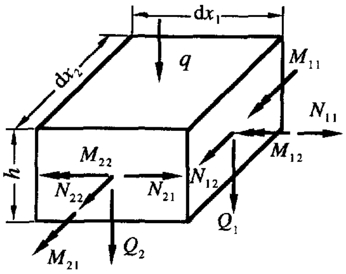

Fig. 2. An infinitesimal free body from a thin plate, showing all the internal resultants on the visible sides. Moment vectors are shown with double arrows.

$$\tau_{\alpha\beta} = D_{\alpha\beta\gamma\delta}(u_{\gamma,\delta} - w_{,\gamma\delta}x_3) - \eta p_f \delta_{\alpha\beta}. \tag{12}$$

Further, in order to express the above relations in global quantities only, we define the stress resultants and the stress moment resultants (shown in Fig. 2) by

$$N_{\alpha\beta} = \int_{-h/2}^{h/2} \tau_{\alpha\beta}\, dx_3, \quad M_{\alpha\beta} = \int_{-h/2}^{h/2} \tau_{\alpha\beta} x_3\, dx_3. \tag{13}$$

Substituting eqn (12) into these expressions, we have the constitutive relations in another form

$$\begin{aligned} N_{\alpha\beta} &= D_{\alpha\beta\gamma\delta} h u_{\gamma,\delta} + \eta \delta_{\alpha\beta} N_p \\ M_{\alpha\beta} &= -D_{\alpha\beta\gamma\delta} I w_{,\gamma\delta} + \eta \delta_{\alpha\beta} M_p \end{aligned} \tag{14a,b}$$

where

$$N_p = -\int_{-h/2}^{h/2} p_f\, dx_3, \quad M_p = -\int_{-h/2}^{h/2} p_f x_3\, dx_3 \tag{15}$$

are pore pressure resultants and pore pressure moment resultants respectively, and $I = h^3/12$. The subscript p, as the subscript f previously introduced, is not an index.

We now consider the equilibrium of an element of the plate (Fig. 2). The equilibrium equations are the same as those for an elastic plate in terms of the resultants and the load, i.e. they are

$$N_{\alpha\beta,\beta} = 0, \quad M_{\alpha\beta,\beta} = Q_\alpha, \quad Q_{\alpha,\alpha} + q = 0 \tag{16}$$

where the shearing forces Q_α (shown in Fig. 2) are the integrals of $\tau_{3\alpha}$ over the thickness of the plate, and q is the distributed normal load which is generally dependent on the coordinates x_1 and x_2 and the time variable. These equations can also be given in terms of the displacements, the pore pressure resultants, and the pore pressure moment resultants by using eqn (14)

$$\begin{aligned} D_{\alpha\beta\gamma\delta} h u_{\gamma,\beta\delta} + \eta N_{p,\alpha} &= 0 \\ -D_{\alpha\beta\gamma\delta} I w_{,\alpha\beta\gamma\delta} + \eta M_{p,\alpha\alpha} + q &= 0. \end{aligned} \tag{17a,b}$$

In summary, we have established two sets of partial differential equations consisting of nine equations: six are given in (14) and three in (17). These reduce to the equations for an elastic plate when $\eta = 0$. Eleven variables, $N_{\alpha\beta}$, N_p, $M_{\alpha\beta}$, M_p, u_α and w, are involved in the nine equations.

In order to find two additional equations, we must consider the other aspect of the physics for the plate problem, i.e. the diffusion within the plate plane. The fluid flow in a poroelastic material is governed by Darcy's law, given as

$$\phi(\dot{\underline{u}}_m - \dot{\underline{u}}_m^f) = \frac{k_m}{\mu_f} p_{f,m} \quad [m = 1, 2, 3; \quad \text{no summation on } m] \tag{18}$$

where the dot implies the derivative with respect to time t. $\underline{u}_m$ and $\underline{u}_m^f$ are the displacements respectively at the solid and the fluid phases in the x_m direction. Generally speaking, they are functions of the coordinates x_1, x_2, x_3 and the time t. μ_f is the fluid viscosity and k_m is the permeability in the x_m direction. This form of Darcy's law is obtained directly from the generalized Darcy's law (Biot, 1962) when only elements on the diagonal of the permeability matrix are non-zero, which is justified for orthotropic materials (which include the material considered here). Further, in our case $k_1 = k_2$ due to the material being isotropic in the $x_1 x_2$ plane and $k_3 = 0$ since the present material does not permit fluid flow in the x_3 direction. Noting that $\varepsilon_{ii} = \underline{u}_{i,i}$ and $\varepsilon_{ii}^f = \underline{u}_{i,i}^f$, the combination of (2) and (18) yields

$$\dot{\zeta} = \frac{k_1}{\mu_f} p_{f,\alpha\alpha} \tag{19}$$

in which ζ can be replaced by (7b), and further the strains in (7b) can be calculated by (11); then Darcy's law takes the form

$$K p_{f,\alpha\alpha} = \dot{p}_f + \lambda \hat{D}(\dot{u}_{\alpha,\alpha} - \dot{w}_{,\alpha\alpha} x_3) \tag{20}$$

where the constants are

$$K = \frac{k_1}{\mu_f \hat{\beta}}, \quad \lambda = \frac{\eta}{\hat{D}\hat{\beta}}. \tag{21}$$

Thus five independent physical parameters, $\hat{D}$, v, λ, η and K are required for defining the material properties of the problem. $\hat{\beta}$ will be considered as a derived parameter hereafter.

After eqn (20) is first integrated over the thickness of the plate, and is then multiplied on both sides by x_3 and integrated over the thickness, we get

$$\begin{aligned} K N_{p,\alpha\alpha} - \dot{N}_p + \lambda \hat{D} h \dot{u}_{\alpha,\alpha} &= 0 \\ K M_{p,\alpha\alpha} - \dot{M}_p - \lambda \hat{D} I \dot{w}_{,\alpha\alpha} &= 0. \end{aligned} \tag{22a,b}$$

These are two additional partial differential equations, and therefore the system of differential equations is complete for the eleven variables.

The geometrical boundary conditions and the load boundary conditions are the same as those for an elastic plate. They are

$$u_\alpha = \overline{u_\alpha}, \quad w = \bar{w}, \quad w_{,r} = \overline{w_{,r}} \tag{23}$$

and

$$N_r = \overline{N_r}, \quad M_n = \overline{M_n}, \quad H_n = Q_n + M_{s,s} = \overline{H_n} \tag{24}$$

where the subscript r takes the value of n or s which denote the normal and tangent of the boundary, respectively. Neither n nor s is to be taken as an index. An over-bar refers to a given function. The pore pressure or its derivatives with respect to the normal are given at the boundaries. In terms of the resultants, for a permeable boundary the diffusion boundary conditions are

$$N_p = \overline{N_p}, \quad M_p = \overline{M_p} \tag{25}$$

and for an impermeable boundary

$$N_{p,n} = 0, \quad M_{p,n} = 0. \tag{26}$$

The quantities in the directions of the normal and tangent of the boundaries can be expressed in terms of the corresponding variables in the coordinate directions and the direction cosines of the boundaries, i.e.

$$N_r = N_{\alpha\beta} n_\alpha r_\beta, \quad M_r = M_{\alpha\beta} n_\alpha r_\beta, \quad Q_n = Q_\alpha n_\alpha \tag{27}$$

which are coordinate transformations (e.g., Washizu, 1982) and

$$w_{,r} = w_{,\alpha} r_\alpha, \quad N_{p,n} = N_{p,\alpha} n_\alpha, \quad M_{p,n} = M_{p,\alpha} n_\alpha \tag{28}$$

which are purely mathematical relations for derivatives. The direction cosines ($r_\alpha = n_\alpha, s_\alpha$) of the normal and tangent of a boundary are defined as

$$n_\alpha = \cos(\mathbf{x}_\alpha, \mathbf{n}), \quad s_\alpha = \cos(\mathbf{x}_\alpha, \mathbf{s}) \tag{29}$$

among which $s_1 = -n_2$ and $s_2 = n_1$.

Finally we consider the initial conditions. Physically the initial values for different variables should be compatible. If the loads are suddenly applied at $t = 0$, the relationship between the pore pressure and the displacements at $t = 0^+$ is determined by requiring $\zeta = 0$ which means there is no instantaneous diffusion immediately after the application of the loads. From eqns (7b) and (11)

$$\hat{\beta} p_f + \eta(u_{\alpha,\alpha} - w_{,\alpha\alpha} x_3) = 0 \quad (t = 0^+). \tag{30}$$

Using (15) this gives

$$N_p - \lambda \hat{D} h u_{\alpha,\alpha} = 0, \quad M_p + \lambda \hat{D} I w_{,\alpha\alpha} = 0 \quad (t = 0^+). \tag{31a,b}$$

Two equivalents of the above can be gotten after eqns (14) and (17) are substituted respectively; they are

$$N_{\alpha\beta} = D^{\text{inst}}_{\alpha\beta\gamma\delta} h u_{\gamma,\delta}, \quad M_{\alpha\beta} = -D^{\text{inst}}_{\alpha\beta\gamma\delta} I w_{,\gamma\delta} \quad (t = 0^+) \tag{32}$$

and

$$D^{\text{inst}}_{\alpha\beta\gamma\delta} h u_{\gamma,\beta\delta} = 0, \quad D^{\text{inst}}_{\alpha\beta\gamma\delta} I w_{,\alpha\beta\gamma\delta} - q = 0 \quad (t = 0^+) \tag{33a,b}$$

where

$$D^{\text{inst}}_{\alpha\beta\gamma\delta} = D_{\alpha\beta\gamma\delta} + \lambda\eta \hat{D} \delta_{\alpha\beta} \delta_{\gamma\delta} \tag{34}$$

is the tensor of the instantaneous stiffness at $t = 0^+$ when the fluid is trapped, since eqns (32) and (33) have the same forms as the parallel equations for an elastic plate. In terms of the Young's modulus and Possion's ratio (note that $\hat{E} = \hat{D}(1 - v^2)$, as discussed earlier), the following are the instantaneous quantities respectively

$$\hat{E}_{\text{inst}} = \frac{\hat{E}(1 + v + 2\lambda\eta)}{(1 + \lambda\eta)(1 + v)}, \quad v_{\text{inst}} = \frac{v + \lambda\eta}{1 + \lambda\eta}. \tag{35}$$

Equation (33b) can be simplified to

$$\hat{D}I(1+\lambda\eta)w_{,\alpha\alpha\beta\beta}-q=0 \quad (t=0^+). \tag{36}$$

Thus, all the necessary equations and conditions for solving the plate problem have been found. If we choose the eleven variables, $N_{\alpha\beta}$, N_p, $M_{\alpha\beta}$, M_p, u_α and w, as unknowns, the governing differential equations are the constitutive relations (14), the equilibrium equations (17) and the motion eqns (22). Together with the geometrical boundary conditions (23), the load boundary conditions (24), the diffusion boundary conditions (25) or (26) and the initial conditions (31) or (32), the problem is defined. Noting that the stretching and bending of the plate are not coupled, there are only six and five simultaneous equations respectively for the stretching and bending. Moreover, if we take only five unknowns, N_p, M_p, u_α and w, when the boundary conditions can be decoupled from $N_{\alpha\beta}$ and $M_{\alpha\beta}$, we will not include eqn (14) in the simultaneous differential equations. Thus there would be only three and two simultaneous equations respectively for the stretching and bending.

3. QUASI-STATIC BENDING

The plate problem has been formulated for both in-plane deformation and bending. The two types of deformation are not coupled and therefore they can be considered separately. The in-plane problem has been considered in the past for the case of plane-strain which is formally identical to the present plane-stress problem. See for instance McNamee *et al.* (1960) and Rajapakse (1993). In the following, we consider bending only.

For the sake of simplicity, the quantities are normalized as follows

$$x_\alpha^*=\frac{x_\alpha}{L_\alpha},\quad t^*=\frac{Kt}{L_1^2},\quad \kappa=\frac{L_1}{L_2},\quad w^*=\frac{w}{h}$$

$$M_{\alpha\beta}^*=\frac{M_{\alpha\beta}L_1^2}{\hat{D}Ih},\quad M_p^*=\frac{M_pL_1^2}{\hat{D}Ih},\quad q^*=\frac{qL_1^4}{\hat{D}Ih} \tag{37}$$

where L_α is, generally speaking, the maximum dimension of the plate in the x_α direction. Then the governing eqns (14b), (17b) and (22b) take the following dimensionless forms respectively

$$\begin{aligned} M_{11}^* &= -(w_{,11}^*+\nu\kappa^2 w_{,22}^*)+\eta M_p^* \\ M_{22}^* &= -(\nu w_{,11}^*+\kappa^2 w_{,22}^*)+\eta M_p^* \\ M_{12}^* &= -(1-\nu)\kappa w_{,12}^* \end{aligned} \tag{38a,b,c}$$

$$\nabla^4 w^*-\eta\nabla^2 M_p^*-q^*=0 \tag{39}$$

$$\nabla^2 M_p^*-\dot{M}_p^*-\lambda\nabla^2\dot{w}^*=0 \tag{40}$$

where the differential operator ∇^2 is defined as

$$\nabla^2=\frac{\partial^2}{\partial x_1^{*2}}+\kappa^2\frac{\partial^2}{\partial x_2^{*2}} \tag{41}$$

and the higher order operators are defined as $\nabla^4=\nabla^2\nabla^2$, $\nabla^6=\nabla^2\nabla^4$, etc. The dimensionless forms of all boundary conditions remain unchanged. However the initial conditions (31b) and (36) should be rewritten as follows

$$M_p^*+\lambda\nabla^2 w^*=0 \quad (t^*=0^+) \tag{42}$$

$$(1+\lambda\eta)\nabla^4 w^*=q^* \quad (t^*=0^+). \tag{43}$$

All quantities discussed in this paper hereafter are normalized except when specifically noted; for convenience, the superscript * will be omitted.

Eliminating M_p from (39) and (40) gives a governing equation for w

$$\nabla^6 w-(1+\lambda\eta)\nabla^4\dot{w} = \nabla^2 q-\dot{q}. \tag{44}$$

Together with the initial condition (43) and the corresponding boundary conditions, it is now possible to obtain some analytical solutions for the deflection w.

In order to simplify the solution procedure, the solution of the drained plate is introduced. It is denoted as $w^E = w|_{p_f=0}$ and can be obtained using the elastic theory of thin plates. In other words, w^E is the poroelastic solution when t approaches infinity if the plate has a permeable boundary so that the fluid will finally be drained. Our solutions will be restricted to the plates subjected to time independent loads, i.e. $q = q(x_1, x_2)$ after $t = 0^+$. Now take the solution in the form

$$w = w^E+\Delta w. \tag{45}$$

The governing equation for Δw can be obtained from (44). Noting that $\nabla^4 w^E = q$, $\dot{w}^E = 0$ and $\dot{q} = 0$, we have

$$\nabla^6(\Delta w) = (1+\lambda\eta)\nabla^4(\Delta\dot{w}). \tag{46}$$

The initial condition (43) is seen to be equivalent to

$$\Delta w|_{t=0^+} = -\frac{\lambda\eta}{1+\lambda\eta}w^E \tag{47}$$

if we note that $w|_{t=0^+} = w^E/(1+\lambda\eta)$. For convenience, we write the boundary conditions for Δw according to the types of the constraints. For a simply-supported and permeable boundary, when no moments are applied at the boundary, the boundary conditions for Δw are

$$\Delta w = 0, \quad \Delta w_{,11} = 0, \quad \Delta w_{,1111} = 0 \tag{48a,b,c}$$

if the boundary is parallel to the x_2-axis. Of these, eqn (48a) is a geometrical boundary condition. Equation (48b) is true because $w_{,11} = 0$ from (38a) using $M_{11} = 0$ (no moment), $w_{,22} = 0$ (any derivative with respect to x_2 is zero) and $M_p = 0$ (a permeable boundary). Finally, observing eqn (40), $\nabla^2 w = 0$ gives $\nabla^2 M_p = 0$, which in turn gives $\nabla^4(\Delta w) = 0$ by (39). Thus we obtain eqn (48c), considering that the partial derivatives with respect to x_2 are zero.

Apparently, eqn (48) can be extended to a boundary with arbitrary direction for a simply-supported and permeable boundary, as long as no moment is applied at the boundary. We have

$$\Delta w = 0, \quad \frac{\partial^2 \Delta w}{\partial n^2} = 0, \quad \frac{\partial^4 \Delta w}{\partial n^4} = 0. \tag{49}$$

For a built-in boundary parallel to the x_2-axis, if it is impermeable, the boundary conditions for Δw are

$$\Delta w = 0, \quad \frac{\partial \Delta w}{\partial x_1} = 0, \quad \frac{\partial^5 \Delta w}{\partial x_1^5} = \lambda\eta \frac{\partial^3 \Delta \dot{w}}{\partial x_1^3} \tag{50}$$

where the last equation is obtained from (39) and (40), noting that $M_{p,1} = 0$, $\dot{w}^E = 0$ and the derivatives of Δw with respect to x_2 are zero. If the direction of the boundary is arbitrary, we write

$$\Delta w = 0, \quad \frac{\partial \Delta w}{\partial n} = 0, \quad \frac{\partial^5 \Delta w}{\partial n^5} = \lambda\eta \frac{\partial^3 \Delta \dot{w}}{\partial n^3}. \tag{51}$$

Assume the solution of (46) in the form

$$\Delta w = f(x_1, x_2) \exp(-c^2 t) \tag{52}$$

where c is a constant. We write the exponent in this form because it must be negative so that Δw approaches zero for long time. The unknown function can then be found from the following differential equation

$$\nabla^6 f + c^2(1+\lambda\eta)\nabla^4 f = 0. \tag{53}$$

Finally we can determine M_p by (39) and (40) as follows

$$\eta \dot{M}_p = \nabla^4(\Delta w) - \lambda\eta\nabla^2 \dot{w}. \tag{54}$$

Alternatively it can be obtained in an integral form

$$\eta M_p = -\lambda\eta\nabla^2 w + \int_{0^+}^{t} \nabla^4\{\Delta w(x_1, x_2, \tau)\}\, d\tau \tag{55}$$

where the values of M_p and $\nabla^2 w$ at $t = 0^+$ have been eliminated by using the initial condition (42).

Solutions for a simply-supported rectangular plate permeable at all boundaries are given in the Appendix.

4. TRANSVERSE VIBRATIONS

The basic equations found in the second section are valid for vibrations of the plate if the inertia forces are included in the loads, i.e. q is replaced by $q - \rho\ddot{w}$ (in dimensional form) where ρ is the mass per unit area of the plate which may be a function of position. Thus, after q in (44) is replaced by $q - \gamma^2\ddot{w}$, we get a governing differential equation for w in dimensionless form

$$\nabla^6 w - (1+\lambda\eta)\nabla^4 \dot{w} = \nabla^2(q - \gamma^2\ddot{w}) - (\dot{q} - \gamma^2 \dddot{w}) \tag{56}$$

where

$$\gamma = K\sqrt{\frac{\rho}{\bar{D}I}} \tag{57}$$

is also a dimensionless quantity. For simplicity in finding solutions, γ is taken to be a constant hereafter.

It is interesting to consider the physical meaning of γ. We know that for a simply-supported rectangular plate, if it is drained, the natural (circular) frequencies are

$$\omega_E^{mn} = (m^2+\kappa^2 n^2)\pi^2 \sqrt{\frac{\hat{D}I}{\rho L_1^4}} \tag{58}$$

where m and n are natural numbers. So $\tau_E^{mn} = 1/\omega_E^{mn}$ can be considered as a characteristic time of the drained plate (the period divided by 2π). On the other hand, consider a rectangular plate with permeable boundaries for which the deflection *is held fixed* after an initial pore pressure occurs, i.e. $\dot{w} = 0$ after $t = t_0$. Then, from (40) we can show that the pore pressure moment decays according to

$$M_p = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} B_{ij}^0 \exp\left[-(i^2+\kappa^2 j^2)\pi^2 t\right] \sin(i\pi x_1)\sin(j\pi x_2) \tag{59}$$

in which the coefficients B_{ij}^0 can be determined by the initial value of M_p at $t = t_0$. If this value is proportional to $\sin(m\pi x_1)\sin(n\pi x_2)$ (here m and n are given integers), the only non-zero coefficient is B_{mn}^0 and

$$\tau_D^{mn} = \frac{L_1^2}{(m^2+\kappa^2 n^2)\pi^2 K} \tag{60}$$

is a measure of the diffusion time of the corresponding bending mode (t has been normalized as in (37)). Then we see that γ is the ratio of τ_E^{mn} and the diffusion time τ_D^{mn} of the same bending mode.

In a manner similar to that by which eqns (49) were derived, we find the following boundary conditions of w for a simply-supported permeable boundary with no boundary moments

$$w = 0, \quad \frac{\partial^2 w}{\partial n^2} = 0, \quad \frac{\partial^4 w}{\partial n^4} = q \tag{61a,b,c}$$

where n again refers to the normal of the boundary. In order to solve eqn (56) initial values of w, $\dot{w}$, and $\ddot{w}$ are required. For a given initial pore pressure moment, the initial condition on the second derivative can be gotten by

$$\ddot{w} = (-\nabla^4 w + \eta\nabla^2 M_p + q)/\gamma^2 \tag{62}$$

which is derived from (39) after the inertia is included. So there are only three independent initial conditions. Alternatively, if the initial deflection is produced by a suddenly applied load, we have the following relation from (43) after the inertia is included

$$\ddot{w} = [q-(1+\lambda\eta)\nabla^4 w]/\gamma^2 \quad (t = 0^+) \tag{63}$$

which is justified at $t = 0^+$ only.

Now consider a rectangular plate with the boundary conditions (61) on all four edges. Taking the solution as a double Fourier sine series

$$w = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \phi_{ij}(t)\sin(i\pi x_1)\sin(j\pi x_2) \tag{64}$$

the boundary conditions (61a,b) are met. (61c) is also satisfied if q equals zero at the boundaries. In case q is not zero at the boundaries, we can mathematically change the values in a small area near the boundaries which physically will not alter the solution. The load can be expanded into a series of the same form

$$q = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} B_{ij}(t) \sin(i\pi x_1) \sin(j\pi x_2) \tag{65}$$

where the coefficients are functions of time

$$B_{ij} = 4 \int_0^1 \int_0^1 q(x, y, t) \sin(i\pi x) \sin(j\pi y) \, \mathrm{d}x \, \mathrm{d}y. \tag{66}$$

Thus the problem now is to determine ϕ_{ij} by the following initial conditions

$$\phi_{ij}(0) = 4 \int_0^1 \int_0^1 w(x, y, 0) \sin(i\pi x) \sin(j\pi y) \, \mathrm{d}x \, \mathrm{d}y$$

$$\dot{\phi}_{ij}(0) = 4 \int_0^1 \int_0^1 \dot{w}(x, y, 0) \sin(i\pi x) \sin(j\pi y) \, \mathrm{d}x \, \mathrm{d}y$$

$$\ddot{\phi}_{ij}(0) = 4 \int_0^1 \int_0^1 \ddot{w}(x, y, 0) \sin(i\pi x) \sin(j\pi y) \, \mathrm{d}x \, \mathrm{d}y \tag{67}$$

and the ordinary differential equations

$$\gamma^2 \frac{\mathrm{d}^3 \phi_{ij}}{\mathrm{d}t^3} + \gamma^2 \alpha_{ij} \frac{\mathrm{d}^2 \phi_{ij}}{\mathrm{d}t^2} + (1 + \lambda\eta) \alpha_{ij}^2 \frac{\mathrm{d}\phi_{ij}}{\mathrm{d}t} + \alpha_{ij}^3 \phi_{ij} = \frac{\mathrm{d}B_{ij}}{\mathrm{d}t} + \alpha_{ij} B_{ij} \tag{68}$$

which is derived from (56) and where we record

$$\alpha_{ij} = (i^2 + \kappa^2 j^2)\pi^2. \tag{69}$$

The homogeneous equation of (68) has fundamental solutions in the form $\exp(\alpha_{ij}\xi_k t)$ where ξ_k $(k = 1, 2, 3)$ are the roots of the cubic equation

$$\xi^3 + \xi^2 + \frac{1 + \lambda\eta}{\gamma^2} \xi + \frac{1}{\gamma^2} = 0. \tag{70}$$

The analogous situations for beams were investigated in Li *et al.* (1996). In the present paper we will solve eqn (68) only for free vibrations, i.e. when there are no applied loads; thus $B_{ij} \equiv 0$. Further we only consider the cases for which oscillatory motion is possible, i.e. when eqn (70) has complex roots. Two constants are introduced as follows

$$\tilde{q} = \frac{2}{27} - \frac{1 + \lambda\eta}{3\gamma^2} + \frac{1}{\gamma^2}, \quad \Delta = \left(\frac{1 + \lambda\eta}{3\gamma^2} - \frac{1}{9}\right)^3 + \left(\frac{\tilde{q}}{2}\right)^2. \tag{71}$$

If $\Delta > 0$ there will be one real root and two conjugate complex roots; these are

$$\xi_1 = -1 - 2\tilde{\alpha}, \quad \xi_{2,3} = \tilde{\alpha} \pm \mathbf{i}\tilde{\beta} \tag{72}$$

where $\mathbf{i} = \sqrt{-1}$ and

$$\tilde{\alpha} = \tfrac{1}{2}\left(\sqrt[3]{\tilde{q}/2+\sqrt{\Delta}} - \sqrt[3]{-\tilde{q}/2+\sqrt{\Delta}}\right) - \tfrac{1}{3}$$

$$\tilde{\beta} = \frac{\sqrt{3}}{2}\left(\sqrt[3]{\tilde{q}/2+\sqrt{\Delta}} + \sqrt[3]{-\tilde{q}/2+\sqrt{\Delta}}\right) \quad (73)$$

are real quantities. Therefore the general solution in real form can be written as

$$\phi_{ij} = \exp(\alpha_{ij}\tilde{\alpha}t)[C_1^{ij}\exp(\alpha_{ij}\xi_0 t) + C_2^{ij}\cos(\alpha_{ij}\tilde{\beta}t) + C_3^{ij}\sin(\alpha_{ij}\tilde{\beta}t)] \quad (74)$$

where

$$\xi_0 = \xi_1 - \tilde{\alpha} = -1 - 3\tilde{\alpha}. \quad (75)$$

Given the initial values as in (67), the constants C_k^{ij} are found to be

$$C_1^{ij} = \frac{(\tilde{\alpha}^2+\tilde{\beta}^2)\alpha_{ij}^2\phi_{ij}(0) - 2\tilde{\alpha}\alpha_{ij}\dot{\phi}_{ij}(0) + \ddot{\phi}_{ij}(0)}{(\xi_0^2+\tilde{\beta}^2)\alpha_{ij}^2}$$

$$C_2^{ij} = \phi_{ij}(0) - C_1^{ij}$$

$$C_3^{ij} = \frac{\dot{\phi}_{ij}(0)}{\tilde{\beta}\alpha_{ij}} - \frac{\tilde{\alpha}\phi_{ij}(0) + \xi_0 C_1^{ij}}{\tilde{\beta}}. \quad (76)$$

We can also extract the long time solutions for harmonic loading problems of the plate, $q(x_1, x_2, t) = \hat{q}(x_1, x_2)\exp(\mathbf{i}\omega t)$. For this purpose, we can write the coefficients in (64) and (65) in the forms

$$\phi_{ij}(t) = w_{ij}\exp(\mathbf{i}\omega t), \quad B_{ij}(t) = b_{ij}\exp(\mathbf{i}\omega t). \quad (77)$$

Then eqn (68) becomes an algebraic equation for given i and j, and we find

$$w_{ij} = \frac{(\alpha_{ij}+\mathbf{i}\omega)b_{ij}}{\alpha_{ij}^3 + \mathbf{i}(1+\lambda\eta)\alpha_{ij}^2\omega - \gamma^2\alpha_{ij}\omega^2 - \mathbf{i}\gamma^2\omega^3}. \quad (78)$$

5. NUMERICAL RESULTS AND DISCUSSION

We first consider the quasi-static problem. In Fig. 3, some typical features of the time dependent problem are shown for a suddenly applied uniform load; the plate is square. $\lambda = \eta = 1$ for this and all quasi-static examples. In this and all subsequent figures x_1 and x_2 are replaced by x and y. Figure 4 compares pore pressure distributions for square and rectangular ($\kappa = L_1/L_2 = 2$) plates subjected to a central point load. Note the shape difference of the distributions in the width and length directions. Figure 5 considers the square plate but with the point load located at the three-quarter point on a symmetry axis. The pore pressure distributions are shown through the point of loading. Note that for the x-traverse the maximum pore pressure is not found at the point of loading for $t > 0$.

We now consider free vibrations. The plate is simply-supported and boundaries are permeable. Two types of initial pore pressure are investigated: (I) no initial pore pressure; and (II) the initial pore pressure is prescribed by (42), i.e. produced by a suddenly applied load. The initial deflection considered is $w = \sin(\pi x)\sin(\pi y)$ and the initial speed $\dot{w}$ is taken to be zero.

As expected, the frequencies increase when κ or $\lambda\eta$ increases (or γ decreases). In the cases shown in Fig. 6, the differences between the curves for initial conditions I and II are not large. However, Fig. 7 shows that there can be significant differences. It is clear that the natural frequencies and $\lambda\eta$ make the differences: the higher the natural frequency, the greater the differences are between the curves for the two types of initial pore pressure if $\lambda\eta$

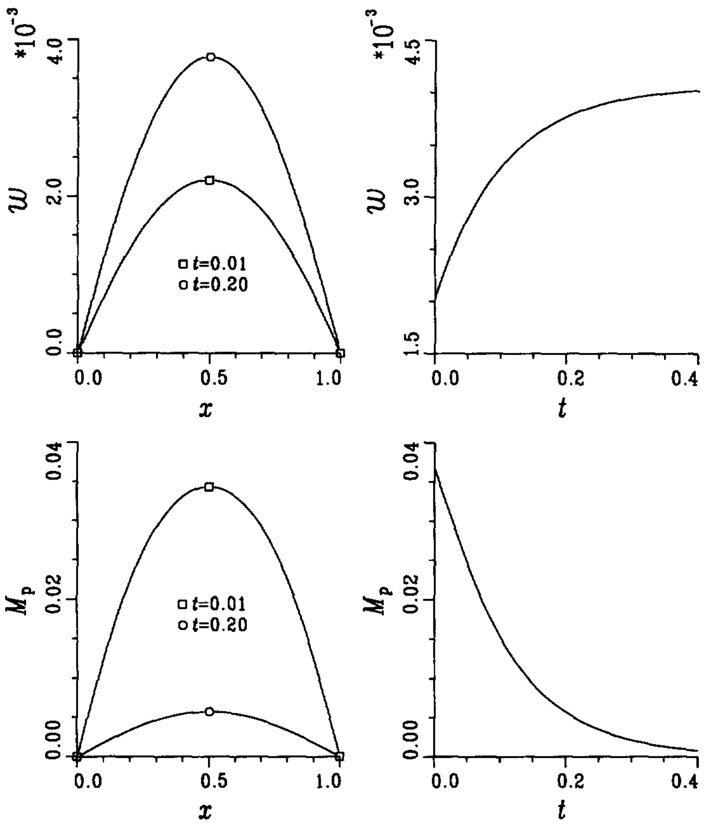

Fig. 3. Deflection and pore pressure vs position and time for a square plate subjected to an uniformly distributed load $q = 1$. $y = 0.5$ is taken for w-x and M_p-x curves, and $x = 0.5$ and $y = 0.5$ is taken for w-t and M_p-t curves.

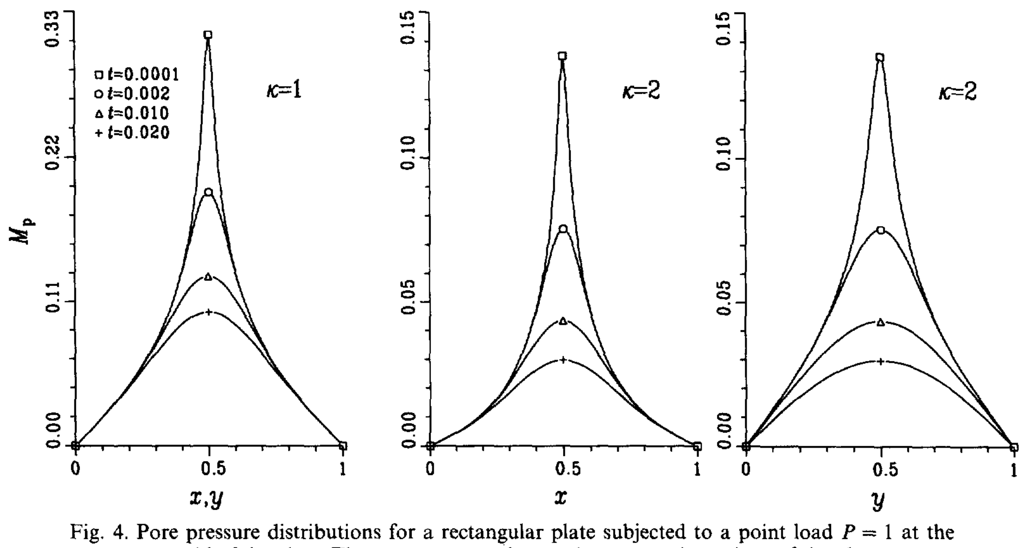

Fig. 4. Pore pressure distributions for a rectangular plate subjected to a point load $P = 1$ at the centroid of the plate. The traverses are taken at the symmetric sections of the plate.

is sufficiently large. When the natural frequency is low, the fluid flows slowly so that little energy is lost in a cycle due to viscosity of the pore fluid. Therefore the behaviour of the system approaches that of an elastic system and the existence of the initial pore pressure is not important. The same is true for a very small $\lambda\eta$. In the opposite situation, the energy dissipation is significant when the fluid flows quickly so that the initial pore pressure plays an important role in the process. When there exists no initial pore pressure (Condition I), there will be no initial impetus for fluid movement (i.e. there are no pressure gradients).

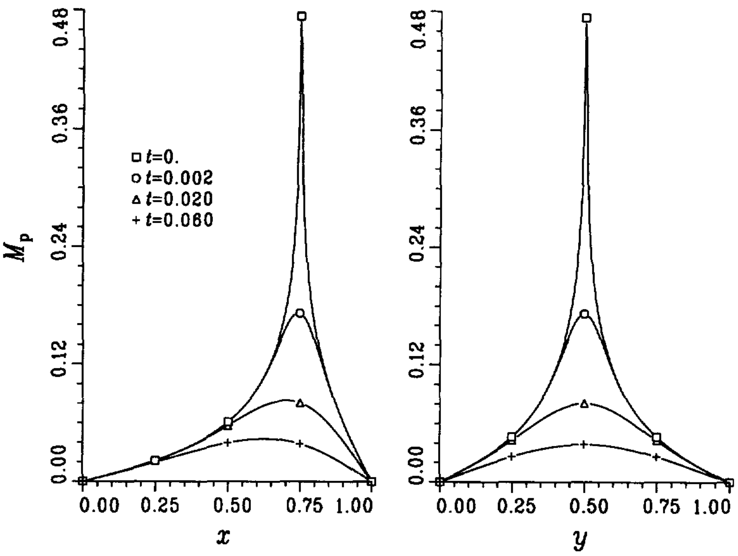

Fig. 5. Pore pressure distributions for a square plate subjected to a point load $P = 1$ at $x = 0.75$ and $y = 0.5$. The traverses are taken through the loading point.

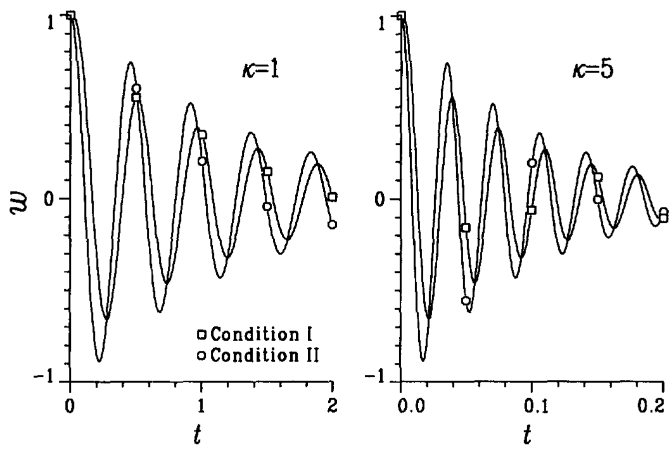

Fig. 6. Decay of the deflection amplitude of a freely vibrating rectangular plate for $\lambda\eta = 0.25$ and $\gamma = 1.5$ for different initial pore pressure conditions.

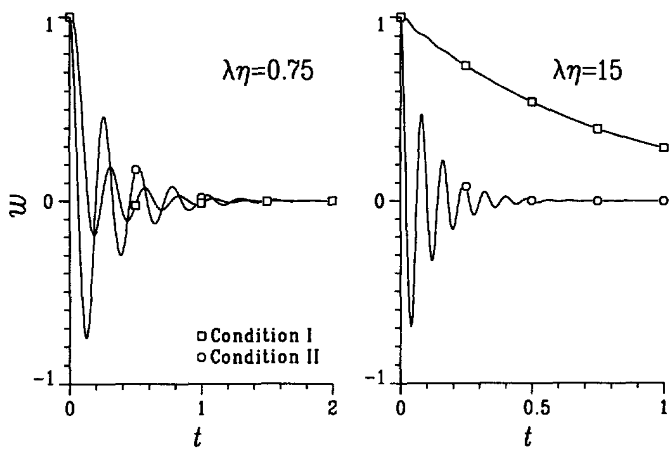

Fig. 7. Decay of the deflection amplitude of a freely vibrating square plate for $\gamma = 1$ for different initial pore pressure conditions.

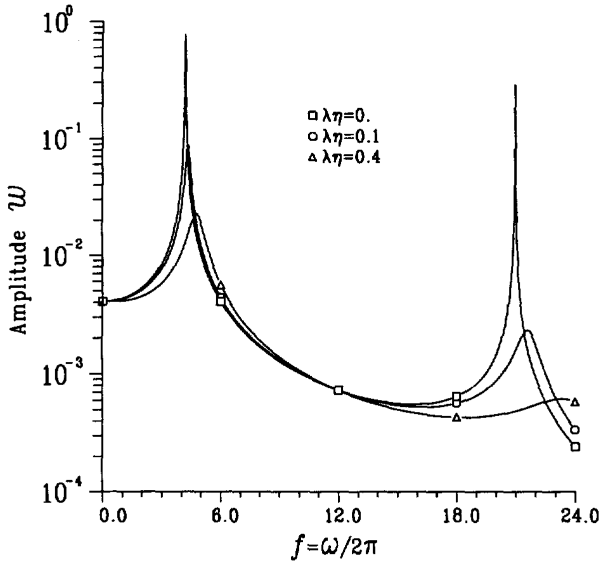

Fig. 8. Resonance areas for a square plate for $\gamma = 0.75$.

The fluid flow, which is then produced by the deflection change, is not able to achieve a high velocity in a short time due to high damping. Hence the plate can only move slowly at first. In other words, the potential energy which is produced by the initial deflection is not able to be converted predominantly into kinetic energy but takes the path of being dissipated finally before the fluid reaches its high velocity. On the other hand, when there exist large pore pressure gradients at the beginning, the fluid flow rapidly reaches a high velocity and the plate vibrates. The energy is being converted quickly among the following: the potential energy of the plate, the potential energy of the fluid produced by the pore pressure gradients, and the kinetic energy of the plate. The energy loss occurs much quicker than for Condition I due to the rapid fluid flow.

Finally we consider forced vibrations. Figure 8 shows the first two resonance regions where ω is the circular frequency of the loading. The load is uniform over the plate surface and of amplitude 1. When $\omega = 0$ we have the long time solutions for the corresponding quasi-static problems (=drained elastic static solutions). $\lambda\eta = 0$ corresponds to the elastic cases. We see that the resonance areas are shifted to the right and the shifts are more distinguishable for higher frequency. On the other hand, the amplitude response is obviously reduced when $\lambda\eta$ increases.

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APPENDIX: QUASI-STATIC SOLUTION FOR A RECTANGULAR PLATE SIMPLY-SUPPORTED AND PERMEABLE AT ALL BOUNDARIES

The boundary conditions for Δw (49) apply to all the boundaries. They are satisfied if we note that the solutions of (53) have the form

$$f_{mn} = A_{mn}\sin(m\pi x_1)\sin(n\pi x_2) \tag{A.1}$$

where m and n are natural numbers. Thus c is determined by (53)

$$c^2 = \frac{(m^2+\kappa^2 n^2)\pi^2}{1+\lambda\eta}. \tag{A.2}$$

Since the governing eqn (46) is homogenous, a possible solution for Δw takes the form

$$\Delta w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\exp(-c^2 t). \tag{A.3}$$

The coefficients A_{mn} can be found by the initial condition (47) in which the elastic solution is given as follows

$$w^E = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A^E_{mn}\sin(m\pi x_1)\sin(n\pi x_2)$$

$$A^E_{mn} = \frac{4\int_0^1\int_0^1 q\sin(m\pi x)\sin(n\pi y)\,\mathrm{d}x\,\mathrm{d}y}{\pi^4(m^2+\kappa^2 n^2)^2} \tag{A.4a,b}$$

where the coefficients can be given in explicit form after the integration is carried out for a particular loading. Those coefficients which are involved in the numerical computations in this paper are given as follows:

For an uniformly distributed load (i.e. q = constant), $A^E_{mn} = 16q/[\pi^6 mn(m^2+\kappa^2n^2)^2]$ if both m and n are odd numbers; $A^E_{mn} = 0$ otherwise.

For a point load P applied at $x_1 = \bar{x}_1$ and $x_2 = \bar{x}_2$, $A^E_{mn} = 4P\sin(m\pi\bar{x}_1)\sin(n\pi\bar{x}_2)/[\pi^4(m^2+\kappa^2n^2)^2]$.

Finally, we can write the deflection as

$$w = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A^E_{mn}\phi_{mn}(t)\sin(m\pi x_1)\sin(n\pi x_2) \tag{A.5}$$

where

$$\phi_{mn}(t) = 1-\frac{\lambda\eta}{1+\lambda\eta}\exp\left(-\frac{(m^2+\kappa^2n^2)\pi^2}{1+\lambda\eta}t\right) \tag{A.6}$$

and A^E_{mn} is given by (A.4b). The pore pressure moment resultant can be extracted from (55)

$$M_p = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}\exp\left(-\frac{(m^2+\kappa^2n^2)\pi^2}{1+\lambda\eta}t\right)\sin(m\pi x_1)\sin(n\pi x_2) \tag{A.7}$$

where

$$B_{mn} = \frac{\lambda\pi^2}{1+\lambda\eta}A^E_{mn}(m^2+\kappa^2n^2) \tag{A.8}$$

and A^E_{mn} is again given by (A.4b).